MTC105

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**B.Sc. Course (CBCS) Ordinance Sem-V**
**EXAMINATION MAY 2023**
**MATHEMATICS - Algebra**

**[Time: 3 Hours]** **[Max. Marks:120]**

**Instructions:** 1. All questions are compulsory; however internal choice is available
2. Figures to the **right** indicate full marks.
3. Use of scientific non-programmable calculators is allowed.
4. Symbols have their usual meanings.

**Q1** Attempt **any five** of the following: **20**

**(a)** An inner automorphism of G,

$$i_g: G \to G,$$

is defined by the map $i_g(x) = gxg^{-1}$ for $g \in G$. Show that $i_g \in$ Aut (G).

**(b)** Find the order of (6, 15, 4) in $\mathbb{Z}_{30} \oplus \mathbb{Z}_{45} \oplus \mathbb{Z}_{24}$.

**(c)** Let G ≈ H. Show that if G is cyclic then so is H.

**(d)** Show that if a is an element of a group G, then $|a| \leq |G|$.

**(e)** Is $\mathbb{Z}_3 \oplus \mathbb{Z}_5$ isomorphic to $\mathbb{Z}_{15}$ ? Justify.

**(f)** For any group G, prove that $\frac{G}{Z(G)}$ is isomorphic to Inn (G).

**(g)** Express $Aut\ (U\ (25))$ in terms of $\mathbb{Z}_m \oplus \mathbb{Z}_n$

**Q2** Attempt **any five** of the following: **20**

**(a)** Let m and n be positive integers and k be the least common multiple of m and n. Show that $m\mathbb{Z} \cap n\mathbb{Z}$.

**(b)** Is $\mathbb{Z} \oplus \mathbb{Z}$ an integral domain? Justify.

**(c)** If $I$ is an ideal of a ring R and 1 belongs to $I$, prove that $I = R$.

**(d)** Prove that $Inn\ (G) \lhd Aut\ (G)$.

**(e)** Prove that the characteristic of an integral domain is 0 or prime.

**(f)** State the following theorems:

i. Factor theorem.

ii. First isomorphism for rings.

**(g)** Let $f\ (x) = 5x^4 + 3x^3 + 1$ and $g(x) = 3x^2 + 2x + 1$ in $\mathbb{Z}_7[x]$.

Determine the quotient and remainder upon dividing $f\ (x)$ by $g(x)$.

**Q3 A)** Attempt **any one** of the following:

**(a)** (i) Prove that an integer $m \in \mathbb{Z}_n$ is a generator of $\mathbb{Z}_n$ if and only if $\gcd(n, m) = 1$. **05**

(ii) Let G be a group. Show that $Z(G) = \cap\{C(a) \mid a \in G\}$. **05**

**(b)** (i) List the elements of $U_4(20)$ and $U_{10}(30)$ .Let n be a positive integer. Prove that $U_k(n)$ is a subgroup of $U(n)$. **05**

(ii) If a and b are any two elements in a group G. Prove that $|a| = |b^{-1}ab|$. **05**

**B)** Let a be an element of order n in a group and let k be a positive integer. Prove that $\langle a^k \rangle = \langle a^{gcd(n,k)} \rangle$ and $|a^k| = \frac{n}{gcd(n,k)}$. **10**

**Q4 A)** Attempt **any one** of the following:

**(a)** (i) Prove that group isomorphism is an equivalence relation. **05**

(ii) Let H and K be subgroups of a finite group G with $H \subseteq K \subseteq G$. Prove that $|G: H| = |G: K||K: H|$. **05**

**(b)** (i) What is the order of each of the following permutations? **04**

$\tau = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 3 & 4 & 5 & 1 & 7 & 8 & 6 \end{bmatrix}$ and $\sigma = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 7 & 6 & 1 & 2 & 3 & 4 & 5 \end{bmatrix}$

(ii) State and prove Orbit-Stabilizer theorem. **06**

**B)** Prove that the order of a permutation of a finite set written in disjoint cycle form is the least common multiple of the lengths of the cycles. **10**

**Q5 A)** Attempt **any one** of the following:

**(a)** Let $n_1, n_2, \ldots\ldots, n_k$ be positive integers. Prove that $\mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2} \oplus \ldots\ldots \oplus \mathbb{Z}_{n_k} \approx \mathbb{Z}_{n_1 n_2 \ldots n_k}$ if and only if $gcd(n_i, n_j) = 1$ for $i \neq j$. **10**

**(b)** Let G be a finite abelian group of order $P^n m$, where p is a prime that does not divide m. Prove that $G = H \times K$, where $H = \{x \in G | x^{p^n} = e\}$ and $K = \{x \in G | x^m = e\}$. **10**

**B)** State and prove Cauchy's theorem for abelian group. **10**

**Q6 A)** Attempt **any one** of the following:

**(a)** (i) Find a positive integer $a$ such that $\langle a \rangle = \langle 6 \rangle \langle 9 \rangle$ and $\langle a \rangle = \langle p \rangle \langle q \rangle$. **05**

(ii) Let R be a ring with unity 1. Show that $S = \{n \cdot 1 | n \in \mathbb{Z}\}$ is a subring of R. **05**

**(b)** (i) Prove or disprove that every ideal of a ring R is the kernel of a ring homomorphism of R. **05**

(ii) Let $F$ be a field, $I$ a non-zero ideal in $F[x]$ and $g(x)$ an element of $F[x]$. Prove that $I = \langle g(x) \rangle$ if and only if $g(x)$ is a non-zero polynomial of minimum degree in $I$. **05**

**B)** Let $R$ be a commutative ring with unity and let $M$ be an ideal of R. Prove that $\frac{R}{M}$ is a field if and only if $M$ is maximal. **10**