MTC106

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B.Sc. COURSE (CBCS) Semester -V
EXAMINATION MAY 2023
MATHEMATICS (ANALYSIS - II)

[Duration : 3 Hours] **[Total Marks : 120]**

Instructions:

(1) All question are compulsory. However internal choice is/are available.
(2) Figures to the right indicate full marks to each question/sub-question.
(3) Use of scientific/non-programmable calculator is allowed.

Q.1 Attempt any five of the following :- **(20)**

(a) Let $\int_a^\infty |f(x)|dx$ be convergent. Prove that $\int_a^\infty f(x)dx$ is convergent.

(b) Discuss the convergence of $\int_0^2 \frac{dx}{x(2-x)}$.

(c) Prove that $\int_a^b \frac{dx}{(b-x)^p}$ is convergent iff $p < 1$.

(d) Show that $\beta(m, n) = \beta(m+1, n) + \beta(m, n+1)$.

(e) Prove that $\Gamma({}^1/_2) = \sqrt{\pi}$.

(f) Let $\sum_{n=0}^\infty a_n x^n$ be convergent for $x_0 \in \mathbb{R}$ and $x_0 \neq 0$.
Prove that $\exists\, M > 0$ such that $|a_n x_0^n| \leq M\ \forall\, n \geq 0$.

(g) Find the interval of convergence of the power series $\sum_{n=0}^\infty \frac{(n+1)^2 x^n}{3^n[(2n)!]^2}$.

Q.2 Attempt any five of the following :- **(20)**

(a) Let $E(x) = \sum_{n=0}^\infty \frac{x^n}{n!}\ \forall\, x \in \mathbb{R}$. Prove that $E(x) > 0\ \forall\, x \in \mathbb{R}$.

(b) Let $C(x) = \sum_{n=0}^\infty \frac{(-1)^n x^{2n}}{(2n)!}\ \forall\, x \in \mathbb{R}$ and $S(x) = \sum_{n=0}^\infty \frac{(-1)^n x^{2n+1}}{(2n+1)!}\ \forall\, x \in \mathbb{R}$. Prove that $C^2(x) + S^2(x) = 1\ \forall\, x \in \mathbb{R}$.

(c) Let $\langle , \rangle$ denote the usual integral inner product on $C[a,b]$ and $\|f\| = \sqrt{\langle f, f\rangle}\ \forall\, f \in C[a,b]$. Prove that $\|f+g\|^2 + \|f-g\|^2 = 2\|f\|^2 + 2\|g\|^2\ \forall\, f, g \in C[a,b]$.

(d) Let $(C[0,1], \langle\ ,\ \rangle)$ be an inner product space over $\mathbb{R}$ with usual integral inner product $\langle , \rangle$. Let $S = \{x+1, x^2+a\}$ be orthogonal subset of $C[0,1]$. Find the value of 'a'.

(e) Obtain the Fourier series of $f(x) = x + \pi\ \forall\, x \in [-\pi, \pi]$.

(f) Let $f: [-\pi, \pi] \longrightarrow \mathbb{R}$ be Riemann integrable and $\frac{a_0}{2} + \sum_{n=1}^\infty [a_n \cos(nx) + b_n \sin(nx)]$

be Fourier series of 'f'. Prove that $\sum_{n=1}^{\infty}(a_n^2 + b_n^2)$ is convergent.

(g) Obtain Fourier sine series of $f(x) = 3x + 4$ on $[0, 1/3]$.

Q.3 (A) Attempt <u>any one</u> of the following :- **(10)**

(a) (i) Let $\lim_{x\to\infty}\left|\frac{f(x)}{g(x)}\right| = l$ be a non-zero finite number. Prove that $\int_a^{\infty}|f(x)|\,dx$ and $\int_a^{\infty}|g(x)|\,dx$ either both converge or both diverge.

(ii) Discuss the convergence of $\int_1^{\infty}\frac{x\,dx}{(1+x)^4}$ and $\int_1^{\infty}\frac{dx}{x(1+\sqrt{x})}$

OR

(b) (i) Let 'f' and 'g' be any two functions defined on [a, b) and $\lim_{t\to b^-}|f(t)| = \infty = \lim_{t\to b^-}|g(t)|$. Let $\exists\, k > 0$ and $t_0 \in [a, b)$ such that $|f(t)| \le k|g(t)|\ \forall\, t \in (t_0, b)$. If $\int_a^b |g(t)|dt$ is convergent at 'b', then prove that $\int_a^b |f(t)|dt$ is convergent at 'b'.

(ii) Discuss the convergence of $\int_0^{\infty}\frac{4+\cos x}{(1+x)\sqrt{x}}\,dx$.

(B) (a) Prove that $\int_0^1 x^{m-1}(1-x)^{n-1}\,dx$ is convergent iff $m > 0$ and $n > 0$. **(6)**

(b) Evaluate $\int_0^{\infty} y^{3/2}\, e^{-y^5}\,dy$ using Gamma function. **(4)**

Q.4 (A) Attempt <u>any one</u> of the following :- **(10)**

(a) (i) Show that $\int_0^1 \frac{x\,dx}{\sqrt{1-x^5}} = \frac{\sqrt{\pi}}{5}\frac{\Gamma(2/5)}{\Gamma(9/10)}$.

(ii) Show that $\left(\int_0^{\infty} xe^{-x^8}dx\right)\left(\int_0^{\infty} x^2 e^{-x^4}dx\right) = \frac{\pi}{16\sqrt{2}}$.

OR

(b) (i) Show that $\int_0^2 x^4(8-x^3)^{-1/3}\,dx = 8\frac{[\Gamma(2/3)]^2}{\Gamma(1/3)}$.

(ii) Show that $\left(\int_0^{\infty}\sqrt{y}\,e^{-y^2}dy\right)\left(\int_0^{\infty}\frac{e^{-y^2}}{\sqrt{y}}dy\right) = \frac{\pi}{2\sqrt{2}}$.

(B) (a) Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series with radius of convergence $R > 0$. Prove that $\sum_{n=0}^{\infty} a_n x^n$ can be differentiated term by term on its interval of convergence $(-R, R)$ and the radius of convergence of power series obtained by differentiating $\sum_{n=0}^{\infty} a_n x^n$ term by term has radius of convergence equal to R. **(5)**

(b) Obtain a power series with interval of convergence as $(-5, 7)$ such that the power series is divergent at -5 and convergent at 7. Justify your claim. **(5)**

Q.5 (A) Attempt <u>any one</u> of the following :- **(10)**

(a) (i) Let $E : \mathbb{R} \to \mathbb{R}^+$ be defined by
$E(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}\ \forall\, x \in \mathbb{R}$ and
$L : \mathbb{R}^+ \to \mathbb{R}$ be such that
$L[E(x)] = x\ \forall\, x \in \mathbb{R}$ and
$E[L(y)] = y\ \forall\, y \in \mathbb{R}^+$. Prove that
$L(pq) = L(p) + L(q)\ \forall\, p, q \in \mathbb{R}^+$.

(ii) State and prove Bessel's inequality in $C[a, b]$ with usual integral inner product.

OR

(b) (i) Let $C(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}\ \forall\, x \in \mathbb{R}$ and $S(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}\ \forall\, x \in \mathbb{R}$. Prove that $C(x + y) = C(x)C(y) - S(x)S(y)\ \forall\, x, y \in \mathbb{R}$.

(ii) Let $\{\phi_n\}$ be an orthonormal sequence in C[a, b] with usual integral inner product $\langle\, , \rangle$. Prove that $\sum_{n=1}^{\infty} |\langle f, \phi_n \rangle|^2 \leq \|f\|^2\ \forall\, f \in C[a, b]$

(B) Show that the set $S = \{\phi_0, \phi_m, \psi_m : m \in \mathbb{N}\}$ is an orthogonal set in $C[{}^{-3}/_2, {}^{3}/_2]$ with respect to the usual integral inner product on $C[{}^{-3}/_2, {}^{3}/_2]$ and hence find the corresponding orthonormal set, where **(10)**

$$\phi_m(x) = \begin{cases} 1 & ; \quad m = 0 \\ \cos\left(\dfrac{2m\pi x}{3}\right) & ; \quad m \in \mathbb{N} \end{cases}$$

and $\psi_m(x) = \sin\left(\frac{2m\pi x}{3}\right)$; $m \in \mathbb{N}$.

(10)

Q.6 (A) Attempt <u>any one</u> of the following :-

(a) (i) State and prove Parseval's relation for Fourier series on $[-\pi, \pi]$.

(ii) Obtain the Fourier series of 'f' given by $f(x) = \begin{cases} x^2 & ; & -\pi \leq x \leq 0 \\ 0 & ; & 0 \leq x \leq \pi \end{cases}$.

OR

(b) (i) Let $f : [-\pi, \pi] \longrightarrow \mathbb{R}$ be Riemann integrable and $\frac{a_0}{2} + \sum_{n=1}^{\infty}[a_n \cos(nx) + b_n \sin(nx)]$ be Fourier series of 'f' on $[-\pi, \pi]$. Let $S_n(x) = \frac{a_0}{2} + [\sum_{k=1}^{n}[a_k \cos(kx) + b_k \sin(kx)] \; \forall \, x \in [-\pi, \pi]$ and $\forall \, n \in \mathbb{N}$. Prove that $\frac{1}{\pi}\int_{-\pi}^{\pi}[S_n(x)]^2 dx = \frac{a_0^2}{2} + \sum_{K=1}^{n}(a_K^2 + b_K^2) \; \forall \, n \in \mathbb{N}$.

(ii) Obtain Fourier sine and cosine series of $f(x) = \pi x - x^2 \; \forall \, x \in [0, \pi]$.

(B) (a) Obtain the Fourier series of $f(x) = x^2 \; \forall \, x \in [-\pi, \pi]$. Hence show that $\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$. (6)

(b) Obtain Fourier cosine series of $f(x) = 2x - x^2$ on $(0, 2)$. (4)