UMTC109

Total No. of Printed Pages: 4

B. Sc. Course (CBCS) Ordinance Sem-VI
EXAMINATION April 2023
MATHEMATICS - COMPLEX ANALYSIS

[Time: 3 Hours] **[Max. Marks: 120]**

Instructions: 1. All questions are compulsory; however internal choice is available.
2. Figures to the right indicate full marks
3. Use of scientific non-programmable calculators is allowed.
4. Symbols have their usual meanings.

Q1 Attempt **any five** of the following: **20 Marks**

a) Find $(2i)^{\frac{1}{2}}$ and express them in rectangular co-ordinates graphically.

b) Sketch the set $|2z+3| > 4$. Is it bounded?

c) Determine whether $f(z) = \frac{z}{\bar{z}}$ is continuous at $z = 0$.

d) Is the function $f(z) = z^2$ differentiable everywhere? Justify.

e) Find the harmonic conjugate of $u(x,y) = 2x(1-y)$.

f) Show that $f(z) = ze^{z}$ is an entire function.

g) Show that $\overline{\cos(iz)} = \cos(i\bar{z}) \ \forall \ z \in \mathbb{C}$

Q2 Attempt **any five** of the following: **20 Marks**

a) Evaluate $\int_0^{\frac{\pi}{6}} e^{2it} dt$.

b) Evaluate $I = \int_c (y - x - i3x^2) dz, z = x + iy$ where C : the line segment AB, A= i ,B =1+i.

c) Evaluate by using an antiderivative, the following integral, where C is any contour

between the indicated limits of integration. $\int_1^{\frac{1}{2}} e^{\pi z} \, dz$.

d) Find the Laurent series Expansion of $f(z) = \frac{1}{z(z-2)^4}$ $in 0 < |z-2| < 2$.

e) Evaluate $\int_C \frac{z^2}{(z-3i)^2}$ dz, C: $|z|=5$. Stae the Result used.

f) Find the image of the region $x > 1, y > 0$ under the transformation $w = \frac{1}{z}$

g) Find Res f(z= -1) for $f(z) = \frac{1}{2+2z+z^2}$

Q3 A) Attempt **any one** (a or b) of the following: **5+5 Marks**

a) i) Find the image of l_1 under the exponential function e: $\mathbb{C} \to \mathbb{C}$, $e(z) = e^z$. where l_1 is the vertical line $x = c$ for a given $c \in \mathbb{R}$.

ii) For any complex number z, define the function cos(z) and hence for any two complex numbers z_1 and z_2 establish the identity

$$\cos(z_1 + z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2.$$

OR

b) i) When do you say a function $f(z) = u(x,y) + i\,v(x,y)$ is analytic? If f(z) is analytic then show that the first order partial derivatives of u(x,y) and v(x,y) exists at a point $z_0 = x_0 + i\,y_0$ and they satisfy the Cauchy Riemann equations.

ii) Discuss the analyticity of the function $f(z) = \frac{1}{z-1}$ at $z=1+i$.

B) Let f: C→C be a differentiable function at $z= z_0$. Show that f is continuous there. Is the converse true? Justify. **10 Marks**

Q4 Attempt **any one** (a or b) of the following: **5+5 Marks**

a) i) State and prove Liouvile's theorem.

ii) Using Cauchy integral formula calculate the integral $\int_C \frac{z\,dz}{(9-z^2)(z+i)}$ where C is the circle $|z| = 2$ described in positive sense.

OR

b) i) State and prove the Fundamental theorem of Algebra.

ii) Apply the Cauchy - Goursat theorem to show that

$$\int_c f(z)dz = 0$$

when the contour C is a unit circle $|z| = 1$ and $f(z) = ze^{-z}$

B) Let a complex valued function f(z) be analytic everywhere within and on a simple closed contour C taken in a positive sense. If z_0 is any point interior to C, then show that **10 Marks**

$$f(z_0) = \frac{1}{2\pi i} \int_c \frac{f(z)dz}{z-z_0}.$$

Q5 Attempt **any one** (a or b) of the following: **5+5 Marks**

a) i) Write the Laurent series expansion of $f(z) = \frac{2z}{1+z^2}$ in the region $1 < |z - i| < 2$

ii) Suppose that $z_n = x_n + i\, y_n$ (n= 1,2,3....) and $z = x + i\, y$.

Then show that $\lim_{n\to\infty} z_n = z$ if and only if $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$

OR

b) i) Use residue theorem to calculate the integral $\int_c \frac{2+3z^2}{(2z-i)(z+1)^2}\, dz$, C is $|z|=2$..

ii) Expand the function $f(z) = \frac{2z+3}{(z-1)(z+5)}$ as a series at z=i.

B) State and prove Cauchy's Residue theorem **10 Marks**

Q6 Attempt any one (a or b) of the following **5+5 Marks**

a) i) Write the Laurent series expansion of $f(z) = z^2\sin(\frac{1}{z})$ when $0 < |z| < \infty$. Hence calculate the residue of f(z) at z = 0 and write the value of the integral $\int_c f(z)dz$, where C the positively oriented unit circle $|z| = 1$.

ii) Find zeros and poles of $(\frac{z+1}{1+z^2})^2$.

OR

b) i) Describe the three types of Isolated singular points by giving examples of each.

ii) Find the fixed points of the transformation $w = \frac{3z-4}{z-1}$.

B) Explain the Residue at infinity for a analytic function.

10 Marks

If a function f is analytic everywhere in the finite plane except for a finite number of singular points interior to a positively oriented simple closed contour C,

then Show that

$$\int_c f(z)dz = 2\pi i \operatorname*{Res}_{z=0}\left[\frac{1}{z^2} f\left(\frac{1}{z}\right)\right]$$