UMTC110

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B. Sc. Course (CBCS) Ordinance (Sem-VI)
EXAMINATION MAY 2023
MATHEMATICS - METRIC SPACES

[Time: 3 Hours] **[Max. Marks:120]**

Instructions:
1) All questions are compulsory (Internal choices are available)
2) Figures to the right indicate full marks.
3) Use to non-programmable calculators is allowed.
4) Symbols have their usual meanings.

Q.I Attempt **any five** of the following: 5x4=20

a) Let *(X, d)* be a metric space and let $d^*: X \times X \to \mathbb{R}$ be defined by $d^*(x,y) = \min\{1, d(x,y)\}$. Then show that d^* is a metric on X.

b) Show that every subset of a describe metric space is both open and closed.

c) Let A= [6, 8] be a subset of the sub space $(Y, d^*)$of the metric space $(\mathbb{R}, d_\mu)$, where d_μ is the usual metric and $Y = [0,4] \cup [5,8]$. Then paid(i) $A^{\circ *} =$ Interior of A is (Y, d^*) and
(ii) $A^{\circ *} =$ Interior of A is $(\mathbb{R}, d_\mu)$, where $d_\mu$is the usual metric on $\mathbb{R}$.

d) Let (X, d) be a metric space and let $x_0, y_0 \in X$. If (x_n) is a sequence in X which converges to x_0, Then show that the sequence $(d(x_{n,}y_0))$ in $(\mathbb{R},\ d_\mu)$ converges to sequence $d(x_n, y_0)$, where d_μ is the usual metric in $\mathbb{R}$.

e) Let (X, d) be a metric space and let (x_n) be a usually Cauchy sequence is X. If (y_n) is another sequence is X, which converges to $y_0 \in X$ and $d(x_{n,}y_n) < {}^1/_n\ \forall n \in \mathbb{N}$, then show that (x_n) also converges to y_0

f) Show that the metric space $(\mathbb{C}, d)$is complete , where
$d(z_1, z_2) = |z_1 - z_2| \ \forall z_1, z_2 \in \mathbb{C}$

g) Let E be a subset of a subspace (Y, d^*) of a metric space (X, d). Then show that E is d^* connected if and only if E is d- connected.

Q.II Attempt **any five** of the following: 5x4=20

a) Show that $(\mathbb{R}, d_\mu)$ is not compact, where d_μ is the usual metric.

b) Let (X, d) be a metric space & A be a connected subset of X. Then shows that $\bar{A}$ is also connected.

c) Show that a closed subset of a compact set is a metric space is compact.

d) Is $([0, 1], d^*)$ a compact metric space? Where d^* is the discrete metric. Justify your answer.

e) Let $f = (\mathbb{R}, d^*) \to (\mathbb{R}, d_\mu)$ be an arbitrary map. Where d^* is the discrete metric on $\mathbb{R}$ and d_μ is the usual metric on $\mathbb{R}$ will f be a continuous map? Justify your answer.

f) Let (X, d_1) & (Y, d_2) be two metric spaces and $f: (X, d_1) \to (Y, d_2)$ be a distance preserving map. i.e., $d_1(x, y) = d_2\big(f(x), f(y)\big)\ \forall\, x, y \in X$. Show that f is continuous.

g) Let A & B be connected subspace of a metric space (X, d) and $A \cap B \neq \phi$. Then show that $A \cup B$ is connected.

Q.III A) Attempt **any one** (a) or (b) form the following.

a)

i) Show that if a set G is open is $(\mathbb{R}^2, d_\infty)$, then it is open in $(\mathbb{R}^2, d_2)$, where $d_\infty(\bar{x}, \bar{y}) = \max\{|x_1 - y_1|, |x_2 - y_2|\}$ and $d_2(\bar{x}, \bar{y}) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$ for all $\bar{x} = (x_1, x_2)$ & $\bar{y} = (y_1, y_2) \in \mathbb{R}^2$ **05**

ii) Let (X, d) be a metric space and A be a subset of X. Then show that $\bar{A} = A$ if is closed, where $\bar{A}$ is the closure of A. **05**

b)

i) Show that if a let G is open in $(\mathbb{R}^2, d_2)$, then it is open in $(\mathbb{R}^2, d_1)$, where $d_1(\bar{x}, \bar{y}) = |x_1 - y_1| + |x_2 - y_2|$ and $d_2(\bar{x}, \bar{y}) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$ for all $\bar{x} = (x_1, x_2)$ & $\bar{y} = (y_1, y_2) \in \mathbb{R}^2$ **05**

ii)

1) Find the boundary of Q in $(\mathbb{R}, d_\mu)$, where d_μ is the usual metric. **2 ½**

2) Is Q dense in $(\mathbb{R}, d^*)$, where d^* is the discrete metric? Justify your answer. **2 ½**

B) Let (X, d) be a complete metric space and Y be a sub space of X. Then show that Y is Complete if and only if Y is closed. **10**

Q.IV A) Attempt any one (a) or (b) from the following.

a)

i) Let (X, d_1)& (Y, d_2) be two metric spaces and let f be a mapping from X into Y, then show that f is continuous on X, if and only if for every $G\ d_2\ open\ in\ Y, f^{-1}(G) is\ d_1$ – open in X. **05**

ii) Let (X, d) be a metric space and D be a dense subset of X. if f is a continuous mapping from X into itself, such that $f(x) = x\ for\ all\ x\epsilon D$, their show that f is Identity on X **05**

b)

i) Let (X, d_1)& (Y, d_2) be two metric spaces and let f be a mapping from X into Y. Then show that f is continuous on X if and only if for each $x\epsilon X$, the inverse image of every d_2 – neighbourhood of f(x) is a d_1- neighborhood of x. **05**

ii) Let (X, d_1)& (Y, d_2) be two metric spaces X be connected and "F" be a continuous mapping from X into Y. then show that f(X) is connected. **05**

B) State and prove the Lebesgue Covering Lemma. **10**

Q. V A) Attempt any one (a) or (b) from the following.

a)

i) Show that every sequentially compact metric space (X, d) is totally bounded. **05**

ii) Show that a subset A of $(\mathbb{R}, d_\mu)$ is compact if and only if A is bounded and closed, where d_μ is the usual metric on $\mathbb{R}$. **05**

b)

i) Let (X, d_1)& (Y, d_2) be two metric spaces, let X be compact metric space and f be a continuous mapping from X into Y. then show that f(x) is compact in Y. **05**

ii) Show that every compact subset A of a metric space (X, d) is closed. **05**

B) Show that a subset E of $(\mathbb{R}, d_\mu)$ is connected if and only if E is an interval, where d_μ is that usual metric. **10**

Q.VI A) Attempt any one (a) or (b) from the following

a)

i) Consider the Non linear space$(C[0,1], ||\ ||_\infty)$. Show that the map $F: (C[0,1], ||\ ||_\infty) \to (\mathbb{R}, d_\mu)$ defined by $F(f) = \int_0^1 f(t)dt$ is continuous on ([0,1], where $d_\mu$is the usual metric on $\mathbb{R}$. **05**

ii) Let D be dense subset in the metric space (X, d_1). Let (Y, d_2) be another metric space and $f: X \longrightarrow Y$ be a continuous onto map. Then show that f(D) is dense in Y. **05**

b)

i) Let (0, 1] and $[1,\infty)$ be subspaces of $(\mathbb{R}, d_\mu)$, where d_μ is the usual metric in $\mathbb{R}$. show that the function $F: (0,1] \longrightarrow [1, \infty)$ defined by $f(x)=\frac{1}{x}\ \forall x \epsilon\ (0,1]$ is a continuous bisection. **05**

ii) Let $F: \mathbb{R} \longrightarrow \mathbb{R}$ be such that $f^{-1}(a, \infty)$ & $f^{-1}(-\infty, b)$ are open in $(\mathbb{R}, d_\mu)$ for any $a, b \epsilon \mathbb{R}$ & d_μ is the usual metric on $\mathbb{R}$. There show that f is continuous on $\mathbb{R}$. **05**

B) Show that a metric space is connected if and only if every continuous function $f: X \longrightarrow \{\pm 1\}$ is a constant function. **10**