UMTE103

Total No. of Printed Pages: 2

B.Sc. Course (CBCS) Ordinance (Sem-VI)
EXAMINATION MAY-2023
MATHEMATICS - NUMBER THEORY

[Time:2 Hours] **[Max. Marks: 80]**

Instructions:
1) All questions are compulsory. However internal choice is/are available.
2) Figures to the right indicate full marks to each question/sub-question.
3) Use of scientific / non-programmable calculator is allowed.

Q1 Attempt any four of the following:- **16**

a) Use Euclidean algorithm to find gcd of 200 and 275. Hence express gcd (200, 275) as 200m +275n for some $m,\ n \in \mathbb{Z}$.

b) State and prove Euclid's lemma.

c) Prove that there infinitely many positive primes.

d) Let a, b $\in \mathbb{Z}$ and n $\in \mathbb{N}$ such that $a \equiv b(mod\ n)$. Prove that $gcd\ (a, n)\ =\ gcd\ (b, n)$.

e) Solve: $14x\ \equiv 10\ (mod\ 8)$.

f) Prove that $\sum_{d/n} \mu(d) = \begin{cases} 1; n = 1 \\ 0; n > 1, \end{cases}$ where n $\in N$.

Q.2 Attempt any four of the following:- **16**

a) Let n = $P_1^{\alpha 1}\ \ P_2^{\ \alpha 2}\ldots\ldots\ P_r^{\ \alpha r}$ be prime factorisation of an integer n > 1. Prove that $\sum_{d/n} \frac{\mu 2(d)}{\tau(d)} = 3^r/2^r$

b) For , $x \in R$, let [x] denote greatest integer less than or equal to 'x' prove that

i) $a \leq b \leq 0 \rightarrow [a][b] \geq [ab]$.

ii) $0 \leq a \leq b\ \rightarrow [a][b] \leq [ab]$.

c) Let 'P' be a prime and $k \in \mathbb{N}$. Prove that $\varphi\ (P^k) = P^k - P^{k-1}$

d) Find the sum of all positive integers less than or equal to 484 and which are relatively prime to 484.

e) Find three different Pythagorean triples, not necessarily primitive, of the form $(16, Y, Z)$.

f) Show that the area of a Pythagorean triangle can never be equal to perfect square.

Q.3 A) Attempt any one of the following:- **06**

i) Let $a, b, c \in \mathbb{Z}$. Prove $\gcd(a, b, c)\ = gcd\ (gcd\ (a, b), c)$

OR

ii) Let a,b $\in \mathbb{N}$, d = $\gcd(a, b)$ and m=$lcm\ (a, b)$. Prove that ab = dm.

B) Let (x_o, y_o) be any solution of linear diphantine equation $ax + by = c$. prove that all solution of $ax + by = c$ are given by $x = x_o + (\frac{b}{d})\,t$; $t \in \mathbb{Z}$, $y = y_o - (\frac{a}{d})\,t$ **06**

where $d = gcd\ (a, b)$

Q.4 A) Attempt any one of the following:- **06**

i) State and prove. Fermat's theorem

OR

ii) Let $a, b \in \mathbb{Z}$ and $n \in \mathbb{N}$ such that $gcd\ (a, n) = 1$ prove that the linear congruence $ax \equiv b$ (mod n) has unique incongruent solution modulo 'n'

B) Solve the following system of linear congruences:- **06**

$X \equiv 4 \pmod 3$
$X \equiv 7 \pmod 4$
$X \equiv 6 \pmod 7$

Q.5 A) Attempt <u>any one</u> of the following:- **06**

i) Let $n = p_1^{\alpha 1}\, p_2^{\alpha 2} \ldots\ldots p_r^{\alpha r}$ be prime factorisation of integer n>1 then , prove that

(I) $\tau(n) = (p_1+1)\,(p_2+1) \ldots\ldots (pr+1)$

(II) $\sigma(n) = \left(\frac{p_1^{\alpha_1+1}-1}{p_1-1}\right) + \left(\frac{p_2^{\alpha_2+1}-1}{p_2-1}\right) \ldots\ldots\ldots \left(\frac{p_r^{\alpha_r+1}-1}{p_r-1}\right)$

OR

ii) Let $n \in N$ and 'p' be a prime. Prove that the exponent of 'p' in the prime factorisation of 'n!' is $\sum_{k=1}^{\infty} [\frac{n}{pk}]$, where $[x]$ denotes greatest integer less than or equal to 'x'

B) Prove that the Euler's function 'φ' is multiplicative. **06**

Q.6 A) Attempt <u>any one</u> of the following:- **06**

.i) Let a, b, c $n \in \mathbb{N}$ such that $\gcd(a, b) = 1$ and $ab = c^n$. Prove that $\exists$ $a_1, b_1 \in \mathbb{N}$ such that $a = a_1^n$ $b = b_1^n$ and $gcd\ (a_1, b_1) = 1$

OR

iii) Prove that the diophantine equation $x^4 - y^4 = z^2$ has no solution in positive integers.

B) let (x, y, z) be a primitive Pythagorean Triple. Prove that $12|xy$ and $60|xyz$. **06**