Paper / Subject Code: MTC106 / Mathematics - Analysis II

MTC 106

Total No. of Printed Pages:3

**B.Sc. (CBCS) (Semester V)**
**EXAMINATION NOVEMBER 2022**
**Mathematics**
**Analysis-II**

[Duration: 3 Hours]                                                [Total Marks :120]

**Instructions:**
1) All questions are compulsory. However, internal choice is/are available.
2) Figures to the right indicate full marks to each question/ Sub-question.
3) Use of scientific/non-programmable calculator is allowed.

1.      Attempt any five of the following: -                                          20
   a) Define, "Improper integral of type-I" and "Improper integral of type-II".
   b) Let $F(t) = \int_a^t f(x)\, dx$ exists $\forall\, t \in (a, b)$. Prove that $\int_a^b f(x)\, dx$ is convergent iff for any
      $\varepsilon > 0, \exists\, \delta > 0$ such that $\left|\int_x^y f(t)\, dt\right| < \varepsilon \forall x, y \in (b - \delta, b)$.
   c) Prove that $\int_a^b \frac{dx}{(x-a)^P}$ is convergent $iff\ P < 1$.
   d) Show that $\beta(m, n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\, dx$.
   e) Prove that $\Gamma(n + 1) = n! \forall n \in \mathbb{N}$.
   f) Express $\int_0^\infty x^3 e^{-4x^4}\, dx$ in terms of gamma function.
   g) Prove that $\int_0^\infty e^{-x^2}\, dx = \frac{\sqrt{\pi}}{2}$.

2.      Attempt any five of the following: -                                          20
   a) Let $\sum_{n=0}^\infty an\ x^n$ be a power series which is convergent for $x_0 \in \mathbb{R}$ and $x_0 \neq 0$. Prove that
      $\sum_{n=0}^\infty an\ y^n$ is absolutely convergent $\forall y$ such that $|y| < |x_0|$.
   b) Find radius of convergence of $\sum_{n=0}^\infty \frac{n!(3x+2)^n}{(n+1)^n 3^n}$.
   c) Let $E: \mathbb{R} \to \mathbb{R}^+$ be a function defined as $E(x) = \sum_{n=0}^\infty \frac{x^n}{n!}\ \forall x \in \mathbb{R}$. Prove that E is strictly
      increasing.
   d) Let $(C[0, 1], \langle, \rangle)$ be an inner product space with the usual integral inner product $\langle, \rangle$ on C
      $[0,1]$ and $\|f\| = \sqrt{\langle f, f \rangle}\ \forall f \in C[0, 1]$. Let $f(t) = 2t^2 - 3\ \forall t \in [0, 1]$ and $g(t) = 4 -$
      $3t\ \forall t \in [0, 1]$. Find $\|f - g\|$.
   e) Let $(C[a, b], \langle, \rangle)$ be an inner product space with the usual integral inner product $\langle, \rangle$ on
      $C[a, b]$ and $\|f\| = \sqrt{\langle f, f \rangle}\ \forall f \in C[a, b]$.
      Prove that $\|f + g\| \le \|f\| + \|g\| \forall f, g \in C[a, b]$
   f) State the Dirichlet conditions for the convergence of the Fourier series of a function 'f'

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on $[-\pi, \pi]$. Hence, redefine $f(x) = \begin{cases} x - \pi; & -\pi < x < 0, \\ \pi - x; & 0 < x < \pi \end{cases}$ to satisfy the Dirichlet conditions.

g) Obtain the Fourier sine series of $f(x) = x^2 + \pi^2 \forall x \in [0, \pi]$.

3. A) Attempt any one of the following: -                                              10
   a) i) Let $a > 0$. Prove that $\int_a^\infty \frac{1}{x^P} dx$ is convergent iff $P > 1$.
      ii) Discuss the convergence of $\int_0^1 \frac{dx}{x^{1/3} (1-x)^{1/2}}$.

                                        OR

   b) i) Let $\int_a^t f(x) \, dx$ and $\int_a^t g(x) \, dx$ both exist $\forall t \in [a, b]$ and $\lim_{t \to b-} \left| \frac{f(t)}{g(t)} \right| = l$ be a non-zero finite number. Prove that $\int_a^b |f(x)| \, dx$ and $\int_a^b |g(x)| \, dx$ both behave alike.
      ii) Discuss the convergence of $\int_0^\infty \frac{1 + \cos^2 x}{x\sqrt{1+x^2}} \, dx$.

   B)   a) Prove that $\int_0^\infty x^{-p}(1 + x)^{-q} \, dx$ is convergent iff $P < 1$ and $p + q > 1$.     6

        b) Prove that $\int_0^1 \frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}} \, dx = |3(m, n)$.                      4

4. A) Attempt any one of the following: -                                              10
   a) i) State and prove Legendre's duplication formula.
      ii) Express $\int_0^1 \frac{dx}{\sqrt{1-x^3}}$ in terms of gamma function.

                                        OR

   b) i) Prove that $\int_0^\infty x^{n-1} e^{-x} \, dx$ converges iff $n > 0$.
      ii) Show that $\left( \int_0^{\pi/2} \sqrt{\cos x} \, dx \right) \left( \int_0^{\pi/2} \frac{1}{\sqrt{\sin x}} \, dx \right) = \pi$.

   B) a) Let $E: \mathbb{R} \to \mathbb{R}^+$ be defined as, $E(x) = \sum_{n=0}^\infty \frac{x^n}{n!} \forall x \in \mathbb{R}$ and $L: \mathbb{R}^+ \to \mathbb{R}$ be a function      such     7
   that $L[E(x)] = x \, \forall \, x \in \mathbb{R}$ and $E[L(y)] = y \, \forall \in \mathbb{R}^+$. Prove that
   $L(p^m) = mL(p) \forall p \in \mathbb{R}^+$ and $\forall m \in \mathbb{R}$.

   b) By integrating suitable power series term by term over a suitable interval show that     3
   $\sum_{n=1}^\infty \frac{1}{(n+1)!} = e - 2$.

5. A) Attempt any one of the following: -                                              10
   a) i) Let $E: \mathbb{R} \to \mathbb{R}^+$ be a function defined as, $E(x) = \sum_{n=0}^\infty \frac{x^n}{n!} \forall x \in \mathbb{R}$. Prove that $E(x + y) = E(x)E(y) \forall x, y \in \mathbb{R}$. Also prove that $E(x)E(-x) = 1 \forall x \in \mathbb{R}$.
      ii) State and prove Parseval's relation in $C[a, b]$ with usual integral inner product.

                                        OR

   b) i) Let $C(x) = \sum_{n=0}^\infty \frac{(-1)^n x^{2n}}{(2n)!} \forall x \in \mathbb{R}$ and $S(x) = \sum_{n=0}^\infty \frac{(-1)^n x^{2n+1}}{(2n+1)!} \forall x \in \mathbb{R}$. Prove that
      $\exists$ , $p \in (0, 2)$ such that $C(p) = 0$ and $S(p) = 1$.
      ii) Let $\{\phi_n\}$ be an orthonormal sequence in $C[a, b]$ with usual integral inner product $\langle , \rangle$

such that $\sum_{n=1}^{\infty} C_n \phi_n$ converges uniformly to '$f$' on [a, b]. Prove that
$C_n = \langle f, \phi_n \rangle \forall n \in \mathbb{N}$.

B) Show that the set $\Phi = \left\{ 1, \cos\left(\frac{5m\pi x}{2}\right), \sin\left(\frac{5m\pi x}{2}\right) : m \in \mathbb{N} \right\}$ is orthogonal in $C\left[\frac{-2}{5}, \frac{2}{5}\right]$ with  **10**

respect to the usual integral inner product on $C\left[\frac{-2}{5}, \frac{2}{5}\right]$. Hence, obtain the corresponding orthonormal set.

6.  A) Attempt any one of the following: -  **10**

a)  i) Let $\frac{a_0}{2} + \sum_{n=1}^{\infty}[a_n \cos(nx) + b_n \sin(nx)]$ coverage uniformly to a bounded function

'$f$' on $[-\pi, \pi]$. Prove that $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx$, $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx$ and $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx$.

ii) obtain the Fourier series of $f(x) = e^x - 1 \forall x \in [0, 2\pi]$.

**OR**

b)  i) Let $f: [-\pi, \pi]$ be a Riemann integrable function and $\frac{a_0}{2} + \sum_{n=1}^{\infty}[a_n \cos(nx) + b_n \sin(nx)]$ be a Fourier series of '$f$' on $[-\pi, \pi]$. Prove that $\frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2) \leq \frac{1}{\pi}\int_{-\pi}^{\pi}[f(x)]^2\, dx$.

ii) Obtain the Fourier series for $f(x) = \begin{cases} 2x - \pi; & -\pi \leq x < 0 \\ \pi - 2x; & 0 \leq x \leq \pi \end{cases}$

B) a) Obtain Fourier series for $f(x) = \frac{(\pi - x)^2}{4} \forall x \in [0, 2\pi]$. Hence show that  **7**

$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}$.

b) Obtain Fourier sine series for $f(x) = 2x - 1 \forall x \in (0,3)$.  **3**