Total No. of Printed Pages:3

B.Sc. (CBCS) (Semester-V)
EXAMINATION NOVEMBER 2022
Mathematics
Algebra

[Duration : Three Hours] **[Total Marks : 120]**

Instructions :
1. All questions are compulsory, however internal choice is available
2. Figures to the **right** indicate full marks.
3. Use of scientific non-programmable calculators is allowed.
4. Symbols have their usual meanings.

1. Attempt **any five** of the following: **20**
   (a) Find all generators of Z_8 and Z_{20}.

   (b) List all elements of $Z_4 \oplus Z_2$.

   (c) Express the following permutations as products of transpositions and identify them as even or odd.
   $$(17254)\,(1423)\,(154632)$$
   and $\begin{bmatrix} 1\,2\,3\,4\,5\,6\,7\,8\,9 \\ 2\,5\,4\,3\,6\,1\,7\,9\,8 \end{bmatrix}$.

   (d) Prove that $U(5)$ is isomorphic to $U(10)$, but $U(12)$ is not. Justify your claim.

   (e) Show that if G is the internal direct product of $H_1, H_2, \ldots, H_n$, and $i \neq j$ with $1 \leq i, j \leq n$, then $H_i \cap H_j = \{e\}$.

   (f) Prove that $\mathbb{Z}/\langle n\rangle \approx \mathbb{Z}_n$.

   (g) State the following theorems:
   i. Fermat's Little theorem
   ii. Orbit-Stabilizer theorem

2. Attempt **any five** of the following: **20**
   (a) Construct a multiplication table for $Z_2[i]$, the ring of Gaussian integers modulo 2. Is this ring a field? Is it an integral domain?

   (b) Let $S = \{a + bi | a, b \in \mathbb{Z}, b \text{ is even}\}$. Show that S is a subring of $\mathbb{Z}[i]$ but not an ideal of $\mathbb{Z}[i]$.

   (c) Let $\mathbb{Z}[\sqrt{2}] = \{a + b\sqrt{2} | a, b \in \mathbb{Z}\}$ and $H = \left\{\begin{bmatrix} a & 2b \\ b & a \end{bmatrix} | a, b \in \mathbb{Z}.\right\}$ Show that $\mathbb{Z}[\sqrt{2}]$ and H are isomorphic as rings.

(d) Let R and S be rings. Prove that $R \oplus S$ is ring isomorphic to $S \oplus R$.

(e) State and prove remaider theorem.

(f) Define the following terms:
i. Characteristic of a ring
ii. Principal ideal domain.

(g) Let $f(x) = 5x^4 + 3x^3 + 1$ and $g(x) = 3x^2 + 2x + 1$ in $\mathbb{Z}_7[x]$. Determine the quotient and remainder upon dividing $f(x)$ by $g(x)$.

3. A) Attempt **any one** of the following:

(a) (i) Prove or disprove the following: **05**
If k is a positive divisor of n, then the number of elements of order k in cyclic group of order n is $\emptyset\ (k)$.

(ii) Let G be a group and let $a \in G$. If $|a| = n$, then prove that $\langle a \rangle = \{a, a^2, a^3, \ldots a^{n-1}, a^n\}$ and $a^i = a^j$ if and only if n divides $i - j$. **05**

(b) (i) If a cyclic group G is generated by an element a of order n, prove that a^m is a generator of G if and only if gcd $(m, n) = 1$. **05**

(ii) Let H be a subset of a group G. Prove that H is a subgroup of G if and only if $H \neq \emptyset$ and whenever $g, h \in H$ then gh^{-1} is in H. **05**

B) Prove that any cyclic group of order n is isomorphic to $(Z_n, +_n)$. **10**

4. A) Attempt **any one** of the following:

(a) (i) Let H be a subgroup of a group G. Prove that any two left cosets of H are either disjoint or identical. **05**

(ii) Suppose that f is an isomorphism from a group G onto a group T. Prove that $f\big(Z(G)\big) = Z\ (T)$. **05**

(b) (i) Let σ and τ be two disjoint cycles in S_n. Prove that $\sigma\tau = \tau\sigma$. **05**
(ii) Prove or disprove that every group of prime order is cyclic. **05**

B) Prove that the set of even permutations in S_n forms a subgroup of S_n and has the order $\frac{n!}{2}$. **10**

5. A) Attempt **any one** of the following:

(a) (i) Let G and H be finite cyclic groups. Prove that $G \oplus H$ is cyclic if and only if $|G|$ and $|H|$ are relatively prime. **05**

(ii) If K is a subgroup of G and N is a normal subgroup of G, prove that $\frac{K}{K \cap N}$ is isomorphic to $\frac{KN}{N}$. **05**

(b) Let G be a finite abelian group of order $p^n m$, where p is a prime that does not divide m. Prove that $G = H \times K$, where $H = \{x \in G | x^{p^n} = e\}$ and $K = \{x \in G | x^m = e\}$. **10**

B) Prove that every group of order p^2, where p is a prime, is isomorphic to Z_{p^2} or $Z_p \oplus Z_p$. **10**

6. A) Attempt **any one** of the following:

(a) (i) Prove or disprove: **05**

Every finite integral domain is a field.

(ii) Let $M_2(\mathbb{Z})$ be the ring of 2×2 matrices over the integers and let **05**

$$R = \left\{ \begin{bmatrix} a & a-b \\ a-b & b \end{bmatrix} \middle| a, b \in \mathbb{Z} \right\}.$$

Prove or disprove that R is subring of $M_2(\mathbb{Z})$.

(b) (i) If F is a field of characteristic p, then prove that F contains a subfield isomorphic to $\mathbb{Z}_p$. **05**

(ii) Let m be a fixed positive integer. For any integer a, let $\bar{a}$ denote a mod m. Show that the mapping $f: \mathbb{Z}[x] \to \mathbb{Z}_m[x]$ given by **05**

$$f(a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0) = \bar{a}_n x^n + \bar{a}_{n-1} x^{n-1} + \cdots + \bar{a}_0$$

is a ring homomorphism.

B) Let R be a commutative ring with unity and let M be an ideal of R. Prove that $\frac{R}{M}$ is a integral domain if and only if M is prime. **10**