CARMEL COLLEGE OF ARTS, SCIENCE AND COMMERCE FOR WOMEN, NUVEM-GOA

SEMESTER END EXAMINATION, JUNE, 2022

Semester: VI of BSC

Course Title: COMPLEX ANALYSIS **Course Code: MTC109**

Total marks: 120 **Date:** **Duration: 3 hours** **Total No of pages: 2**

Instructions: *1. All questions are compulsory. However internal choice is/are available.*
2. Figures to the right indicate maximum marks to each question/sub question.
3. Use of non-programmable calculator is allowed.

Q.1. Answer <u>any five</u> of the following: **(20)**

a) Use exponential form of complex number to show that $\frac{\sqrt{3}+i}{\sqrt{3}-i} = \frac{1}{2}(1+\sqrt{3}i)$.

b) Define conjugate of a complex number $z = x + iy$ and show that $\overline{z_1 . z_2} = \overline{z_1} . \overline{z_2}$ for any two complex numbers z_1 and z_2.

c) Use the definition of limits of function to show that $\lim\limits_{z \to z_0} 2z^2 + 1 = 2{z_0}^2 + 1$.

d) Find $f'(z)$; where $f(z) = \frac{(1+z^2)^4}{z^2}$ $(z \neq 0)$.

e) Find the Principal value of $(1-i)^{4i}$.

f) Evaluate $\int_C \overline{z}\, dz$ from $z = 0$ to $z = 1 + i$ along the curve C given by $z = t^2 + it$.

g) Evaluate $\int_1^2 \left(\frac{1}{t} - it\right) dt$.

Q.2. Answer <u>any five</u> of the following: **(20)**

a) Evaluate the integral $\int_C \frac{1}{z}\, dz$ where C is the semi-circle $z = re^{i\theta}$, $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$.

b) Define the following terms:
i. Singularity
ii. Isolated singularity
iii. Pole of order m

c) Expand the function $f(z) = \sin z$ in Taylor's series about the point $z = {}^{\pi}/_{4}$.

d) State and prove Cauchy's inequality.

e) State and prove Liouville's Theorem.

f) Expand the function $f(z) = \frac{1-\cos z}{z}$ in Laurent's series about the point $z = 0$.

g) Calculate the residue of the function $f(z) = \frac{1}{z(z-2)^4}$ at the point $z = 2$.

Q.3. A. Answer <u>any one</u> of the following: **(10)**

i. When do you say a function $f(z) = u(x,y) + i\, v(x,y)$ is analytic? If $f(z)$ is analytic then show that the first order partial derivatives of $u(x,y)$ and $v(x,y)$ exists at a point $z_0 = x_0 + iy_0$ and they satisfy Cauchy-Reimann equations.

ii. Discuss the analyticity of the following functions.
a. $f(z) = iz + 2$
b. $f(z) = e^{-z}$
c. $f(z) = |z|^2$

B. Show that $u(x,y) = 2x - x^3 + 3xy^2$ is harmonic function. **(10)**
Determine its harmonic conjugate and hence find the corresponding analytic function.

Q.4. A. Answer <u>any one</u> of the following: **(10)**

i. For a non-zero complex number $z = x + iy$, define the exponential function e^z and prove the following identities.

a. $e^{z_1}e^{z_2} = e^{z_1+z_2}$ b. $\frac{e^{z_1}}{e^{z_2}} = e^{z_1-z_2}$ c. $e^{z+2\pi i} = e^z$

d. $|e^z| = e^x$ where z, z_1 and z_2 are complex numbers.

ii. For any complex number $z = x + iy$ define the $\sin hz$ and hence prove that

a. $\sinh(z_1 + z_2) = \sinh z_1 \cosh z_2 + \cosh z_1 \sinh z_2$

b. $|\sinh z|^2 = \sinh^2 x + \sin^2 y$

B. Find the six roots in rectangular coordinates of the equation $z^6 - 8 = 0$. **(10)**

Q.5. A. Answer <u>any one</u> of the following: **(10)**

i. Let a complex valued function $f(z)$ be analytic everywhere on and within a simple closed contour C taken in the positive sense.

If z_0 is any interior point in C. Then show that $f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-z_0}dz$

ii. Let a complex valued function $f(z)$ be analytic in a simple connected domain D. Suppose that the derivative of $f(z)$ is continuous in D. Show that for every simple closed contour C in D, $\int_C f(z)\, dz = 0$.

B. Use Cauchy's integral formula to evaluate the following integral. **(10)**

a. $\int_C \frac{e^{2z}}{(z-3i)^2}dz$ where C is the circle $|z| = 5$

b. $\int_C \frac{z+1}{z^4+2iz^3}dz$ where C is the circle $|z| = 1$

Q.6. A. Answer <u>any one</u> of the following: **(10)**

i. Define residue of a function at an isolated singular point. Show how it can be used to find the contour integral of a function enclosing an isolated singularity, hence State and prove Cauchy's Residue theorem.

ii. Expand the function $f(z) = \frac{1}{z(1-z)}$ as a Laurent's series, valid in the region

a. $1 < |z + 1| < 2$ b. $|z + 1| > 2$

B. Use Cauchy's residue theorem to evaluate the integrals of the following function around a circle $|z| = 3$ in the positive direction. **(10)**

a. $f(z) = \frac{5z-2}{z^2-z}$ b. $f(z) = \frac{z+1}{z^2-2z}$
