CARMEL COLLEGE OF ARTS SCIENCE AND COMMERCE FOR WOMEN, NUVEM-GOA

B.Sc. CBCS Semester V Examination January 2021

Subject : Mathematics **Paper name and code: Combinatorics (MTE102)**

Total marks: 80 **Duration:2hours** **Total no. of pages: 2**

Instructions: *i) All questions are compulsory*
ii) Figures to the right indicate full marks
iii) Use of non programmable calculators is allowed.

I) Answer <u>any four</u> of the following questions. (4x4=16)

1) Given 9 integers from the set $\{1,2,3,\ldots,16\}$, show that there will always be two among the selected integers whose greatest common divisor is 1.
2) Prove that there exists a positive integer n such that $44^n - 1$ is divisible by 7.
3) Let the sequence $\{a_n\}$ be such that $a_0 = 0$ and $a_{n+1} = a_0 + a_1 + a_2 + \cdots + a_n + n + 1 \ \forall\, n \in \mathbb{N}$. Show that the equality $a_n = 2^n - 1$ holds $\forall\, n$.
4) How many permutations of [n] have 3 3-cycles, 2 2-cycles and 2 1-cycles?
5) State and prove the principle of weak induction.
6) For all positive integers n and k (where $k \leq n$), give a combinatorial proof for the following fact.
$$\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$$

II) Answer <u>any four</u> of the following questions. (4x4=16)

1. How many compositions does the integer 15 have whose first part is 1?
2. For all non-negative integers n, show that $B(n+1) = \sum_{i=0}^{n} \binom{n}{i} B(i)$(where B(k) denotes the k^{th} Bell number).
3. How many three-digit positive integers contain two but not three different digits?
4. Show that the number of n-permutations having only one cycle is equal to (n-1)!
5. State and prove the generalized pigeonhole principle.
6. How many subsets does [n] have that contain exactly one of the elements 1 and 2?

III) Answer the following questions. (6x2=12)

A) Answer any one of the following.

i) Show that the total number of subsets of an n-element set is equal to 2^n.

ii) Show that the number of k-element multisets whose elements all belong to [n] is $\binom{n+k-1}{k}$?

B) Show that for all positive integers $k \leq n$
$$S(n,k) = S(n-1,k-1) + k\,S(n-1,k)$$
Where S(n,k) denotes the number of partitions of [n] in k parts.

IV) Answer the following questions. (6x2=12)

A) Answer any one of the following.

i) Show that for all non-negative integers n and $a_1, a_2, \ldots, a_k$ such that $n = \sum_{i=1}^{k} a_i$, the following equality holds
$$\binom{n}{a_1, a_2, \ldots, a_k} = \binom{n}{a_1} \ldots \binom{n - a_1 - \cdots - a_i}{a_{i+1}} \ldots \binom{n - a_1 - a_2 - \cdots - a_{k-1}}{a_k}$$

ii) For all positive integers n, show that the following equality holds.

$$2^{n-2} \times n \times (n-1) = \sum_{k=2}^{n} k(k-1)\binom{n}{k}$$

B) If $a_{n+1} = (n+1)(a_n - n + 1)$ for $n > 0$ and $a_0 = 1$, then find a closed formula for a_n.

V) Answer the following questions. (6x2=12)

A) Answer any one of the following.

i) Show that the number of weak compositions of n into k parts is equal to $\binom{n+k-1}{k-1}$, where n and k are positive integers.

ii) Let $A_1, A_2, \dots, A_n$ be finite sets. Then show that

$$|A_1 \cup A_2 \cup A_3 \cup \dots \cup A_n| = \sum_{j=1}^{n} (-1)^{j-1} \sum_{i_1, i_2, \dots, i_j} A_{i_1} \cap A_{i_2} \cap \dots \cap A_{i_j}$$

B) Let $c(n, k)$ denote the number of n-permutations with k cycles for positive integers n and k with $n \geq k$. Then show that

$$c(n, k) = c(n-1, k-1) + (n-1)c(n-1, k)$$

VI) Answer the following questions. (6x2=12)

A) Answer any one of the following.

i) Let $q(n)$ be the number of partitions of n in which each part is at least 2. Then show that $q(n) = p(n) - p(n-1)$ for all positive integers $n \geq 2$.

ii) The frog population of an infinitely large lake grows fourfold each year. On the first day of each year 100 frogs are taken out of the lake and shipped to another lake. Assuming that there were 50 frogs originally, how many frogs will be there in n years?

B) A football coach has n players to work with at today's practice. First, he splits them into two groups, and asks the members of each group to form line. Then he asks the members of the first group to take on an orange, white, or a blue T-shirt. Members of the other group keep their red T-shirt. In how many different ways can all this happen?

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